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# Secondary flow in a Hele-Shaw cell

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Riegels (1938) investigated the breakdown of Hele-Shaw flow in a Hele-Shaw cell with unusually large separation distance $2h^*$ between the walls. A theoretical outer expansion for the velocity was constructed in the case where the obstacle is a circular cylinder, using an intuitive inner boundary condition that seems to be correct in the limit $h^* \to 0$, but without explicit matching with the inner expansion.

An inner expansion has now been found, and it shows that the solution in the inner layer forces terms into the outer expansion that are larger than those found by Riegels whenever $h^*$ is finite and not zero.

## 1. Introduction

In 1897 Hele-Shaw, who was making an experimental investigation into factors that influence boundary-layer thickness in the steady flow of a viscous liquid past cylindrical obstacles which have been confined between parallel plane plates, discovered that at very narrow separations of the plates the flow is laminar, and the streamlines as indicated by filaments of coloured dye closely resemble the theoretical streamlines in two-dimensional irrotational flow of an ideal non-viscous liquid past an infinite cylinder having the same cross-section, except in a boundary layer whose thickness is about the same as the distance separating the plates (Hele-Shaw 1897, 1898*a*, *b*). This may be called the *Hele-Shaw Effect.*

Stokes (1898) assuming that the flow is slow enough for inertia terms in the equations of motion to be neglected, and that the velocity component perpendicular to the confining walls is small compared with components in the central plane, showed that for a fixed Cartesian co-ordinate frame in which the walls have equations $z = \pm h^*$ ($h^*$ small), derivatives $\partial^2/\partial z^2$ will be much larger than $\partial^2/\partial x^2$ or $\partial^2/\partial y^2$, and therefore the equations of motion, to leading order, are

$$\left.\begin{aligned} &\text{(i)}\quad \mu\frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x}, &&\text{(ii)}\quad \mu\frac{\partial^2 v}{\partial z^2} = \frac{\partial p}{\partial y},\\ &\text{(iii)}\quad 0 = \frac{\partial p}{\partial z}, &&\text{(iv)}\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\quad \left[+\frac{\partial w}{\partial z}\right] = 0,\end{aligned}\right\} \tag{1.1}$$

where $u$, $v$, $w$ are components of velocity in the $x$, $y$ and $z$ directions respectively, $p$ is the dynamic pressure and $\mu$ the coefficient of shear viscosity.

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The term $\partial w/\partial z$ is in square brackets here because it was omitted by Stokes: however, it is better to retain it since the largeness of the derivative $\partial/\partial z$ may balance the smallness of $w$.

Since $p$ is independent of $z$ by (1.1, iii), the first two equations can be integrated at once. Using the boundary conditions $u = v = 0$ at $z = \pm h^*$, we find

$$\text{(i)} \quad u = \frac{-1}{2\mu}(h^{*2}-z^2)\frac{\partial p}{\partial x}, \quad \text{(ii)} \quad v = \frac{-1}{2\mu}(h^{*2}-z^2)\frac{\partial p}{\partial y}. \tag{1.2}$$

If these values are now substituted into the continuity equation, then bearing in mind that $w$ ought to be an odd function of $z$, there results

$$\text{(iii)} \quad w = \frac{1}{2\mu}(h^{*2}z - \tfrac{1}{3}z^3)\,\Delta p(x,y), \tag{1.2}$$

where
$$\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

This confirms that $w$ is small compared with $u$ and $v$ provided $h^*$ is small. However, when we put $z = \pm h^*$ and try to apply the condition $w = 0$ for $z = \pm h^*$, we find that the coefficient of $\Delta p(x,y)$ cannot vanish, and therefore we must have

$$\Delta p = \frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2} \equiv 0, \tag{1.3}$$

giving $w \equiv 0$ for all $z$ in $-h^* \leqslant z \leqslant h^*$, which indicates that $\partial w/\partial z$ can in fact be omitted from (1.1, iv) without imposing a different form on the final solution.

Thus $u:v:w = -\partial p/\partial x : -\partial p/\partial y : 0$, independent of $z$, and therefore the streamline shapes will be the same in all planes parallel to the walls, and these shapes will in fact be those of streamlines in plane potential flow of an ideal liquid past an obstacle having the same cross-section, in which it is clearly necessary that the obstacle boundary should occur as a streamline (i.e. the normal component of velocity should vanish at the obstacle for each $z$). The latter is a sufficient condition to determine $p$ apart from a constant factor giving the speed at which liquid enters the cell in the central plane.

Since the flow is viscous, the tangential velocity components ought also to vanish at the obstacle, but the freedom to meet this condition was lost in obtaining (1.1), where terms in $\mu\,\partial^2/\partial x^2$, $\mu\,\partial^2/\partial y^2$ occurring in the viscous forces were neglected, and consequently the order of the equations of motion in the variables $x$ and $y$ was depressed.

Stokes noted this difficulty and pointed out that equations (1.1) are not valid near the obstacle because in this region the relation

$$\frac{\partial^2}{\partial z^2} \gg \Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

is not true. The flow régime in a Hele-Shaw cell is thus divided into two regions characterized by the relative magnitudes of $\partial^2/\partial z^2$ and $\Delta$: an outer region where $\partial^2/\partial z^2 \gg \Delta$ and an inner, boundary layer where $\Delta = O(\partial^2/\partial z^2)$.

The form of (1.2) suggests that in the outer region $\partial/\partial z$ is $O(1/h^*)$ whilst $\partial/\partial x$ and $\partial/\partial y$ are $O(1)$, and therefore in the inner region both $\partial^2/\partial z^2$ and $\Delta$ are $O(1/h^{*2})$, and this in turn suggests that the inner region must be of thickness $O(h^*)$. These intuitive ideas are confirmed in the solution we shall obtain.

In Hele-Shaw's original experiments a suitable value of $h^*$ was found by trial, and in a typical case the separation of the walls would be rather less than $\frac{1}{2}$ mm. Riegels (1938) examined the effect of choosing separations of 1–2 mm in a large Hele-Shaw cell containing a circular obstacle, with the object of finding a critical dimensionless parameter whose value should predict for any proposed experimental conditions whether a valid Hele-Shaw effect could be obtained or not.

By substituting the Stokes solution in the hitherto neglected inertia terms of the incompressible Navier–Stokes equations, still supposing that $\partial^2/\partial z^2 \gg \Delta$, Riegels found an improved approximation to the outer velocity through which each of the variables $u$, $v$, $w/h^*$ gains an extra term in the form of a factor

$$\Lambda = \frac{\rho Q a}{\mu}\left[\frac{h^*}{a}\right]^2 = \mathscr{R}h^2, \quad \text{say,}$$

multiplying functions of $x$, $y$ and $z/h^*$ that are $O(1)$ compared with $h^*$, where $Q$ is the speed of liquid entering the cell in the central plane, $\rho$ is its density and $a$ is the radius of the obstacle. Hence $\mathscr{R}$ is a Reynolds number based on the flow in the central plane and $h = h^*/a$ gives a scaling of the separation distance to the dimensions of the obstacle.

Experiments were then performed in which it appeared that the streamline shapes become distorted whenever $\Lambda$ is close to or larger than unity, and on this evidence it was assumed that a solution of the full equations of motion could be obtained in the outer region in the form of a series expansion in powers of $\Lambda$:

$$\mathbf{q}^* = (u, v, w/h^*) = \mathbf{q}_0^*(x, y, z/h^*) + \Lambda\mathbf{q}_1^*(x, y, z/h^*) + \Lambda^2\mathbf{q}_2^* + \ldots, \tag{1.4}$$

where $\mathbf{q}_0^*$ is the Stokes solution and $\mathbf{q}_1^*$ is the improvement described above.

Riegels found, however, that whereas it is possible to determine $\mathbf{q}_0^*$ so that the normal component vanishes on the inner boundary, although tangential components remain finite, it is impossible to determine $\mathbf{q}_1^*$ to make *any* component vanish for all $z$ in $-h^* \leqslant z \leqslant h^*$.

The outer solution is not to be judged defective on this account, for it is not generally a property of outer perturbation solutions that they will meet any of the conditions at the inner boundaries. However, the existence of a physically obvious inner condition for the Stokes leading term $\mathbf{q}_0^*$ enabled this term to be evaluated explicitly without reference to the inner solution of the same order, and if a similar condition could be seen to hold at order $\Lambda$ then it might be hoped that $\mathbf{q}_1^*$ could likewise be fully determined even if the inner equations should prove too difficult to solve to the order which is required to determine $\mathbf{q}_1^*$ by matching.

In fact Riegels did attempt an inner solution and was able to produce an inner tangential velocity term of $O(1)$ which matches $\mathbf{q}_0^*$ when $(x^2+y^2)^{\frac{1}{2}} - a = O(h^*)$ and vanishes for $x^2+y^2 = a^2$, but if higher orders were attempted then apparently the attempt was not successful.

In the absence of an inner solution to $O(\Lambda)$ it was vital to have a plausible inner condition to impose on $\mathbf{q}_1^*$ in order to determine arbitrary quantities appearing in it, and the one Riegels used was

$$\int_{-h^*}^{h^*} \mathbf{q}^*.\hat{\mathbf{n}}\,dz = 0, \tag{1.5}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the surface of the obstacle. Under this condition the obstacle becomes a streamline of the $z$-averaged flow.

Riegels's solution for $\mathbf{q}_0^* + \Lambda \mathbf{q}_1^*$ and $p$ is most conveniently expressed in cylindrical polar variables $(r, \theta, z)$ where $x = r\cos\theta, y = r\sin\theta$. If the corresponding velocity components are $(\overline{u}, \overline{v}, w)$ then these have the following values in Riegels's solution:

$$\left.\begin{aligned}
&\text{(i)} & \overline{u}/Q &= \left(1-\frac{z^2}{h^{*2}}\right)\left(1-\frac{a^2}{r^2}\right)\cos\theta + \Lambda H\left(\frac{z}{h^*}\right)\left(\frac{a^3\cos 2\theta}{r^3}-\frac{a^5}{r^5}\right),\\
&\text{(ii)} & \overline{v}/Q &= \left(1-\frac{z^2}{h^{*2}}\right)\left(1+\frac{a^2}{r^2}\right)\sin\theta + \Lambda H\left(\frac{z}{h^*}\right)\frac{a^3\sin 2\theta}{r^3},\\
&\text{(iii)} & w/Qh^* &= -\frac{4a^5\Lambda}{r^6}\int_0^{z/h^*} H(t)\,dt,\\
&\text{(iv)} & \Lambda p/\rho Q^2 &= -2\left(r+\frac{a^2}{r}\right)\cos\theta - \tfrac{12}{35}\Lambda\left(1-\frac{2a^2}{r^2}\cos 2\theta+\frac{a^4}{r^4}\right),
\end{aligned}\right\} \tag{1.6}$$

where $H(t) = t^6/15 - t^4/3 + 11t^2/35 - 1/21$.

The terms involving $\Lambda$ in (1.6) may be thought of as a secondary flow superposed on the Stokes solution $\Lambda = 0$. They are not symmetric about the line $\theta = \frac{1}{2}\pi$, and streamlines are displaced further on the downstream side. The polynomial $H(t)$ has two zeros in the range $0 \leqslant t \leqslant 1$, one at $t = 0{\cdot}433$ and one at $t = 1{\cdot}0$. In this range $H(t)$ lies between values $-0{\cdot}006$ at $t = 0$ and $0{\cdot}004$ at $t = 0{\cdot}75$, so the extremes of displacement of the streamlines from the Stokes flow will occur (in opposite senses) in the planes whose equations are $z = 0$ and $z = \pm 0{\cdot}75h^*$ respectively.

Riegels plotted streamlines in these two planes for $\Lambda = 4{\cdot}0$ starting from equally spaced points far upstream. Theory was then compared with experiment by superposing these drawings on a photograph of an actual flow with $\Lambda = 4{\cdot}0$. Although the agreement is good in the main, there appears to be an unexplained boundary-layer separation about $\theta = 60^\circ$, and it seems probable that a value as large as $\Lambda = 4{\cdot}0$ lies outside the range of validity of the series (1.4).

In general Hele-Shaw flow should be a two-parameter problem, for $\mathscr{R}$ and $h$ are independently variable. However, the outer series (1.4) proposed by Riegels contains only one parameter $\Lambda = \mathscr{R}h^2$, and accordingly corresponds to the limit $h \to 0$ keeping $\mathscr{R}h^2$ finite, which has been taken at the stage of the outer equations of motion before they have been solved.

Although in this limit the thickness $\delta$ of the boundary layer goes to zero, the inner solution itself still stands: and if we write $r/a = 1 + hX$, in the expectation discussed above that $\delta = O(h)$, then the outer solution is still bound at $r = a$ to match the limiting values of the inner solution as $X \to \infty$. It is still prevented by the depression of order noted above from meeting any other conditions at $r = a$.

Now the inner solution is primarily an expansion in $\delta$, i.e. in $h$, and for this reason a matching even with (1.4) requires the outer solution to be treated as an expansion in $h$; and if the limit $h \to 0$ is to be taken this must happen after the matching has been done.

A new analysis of the Hele-Shaw problem for the circular cylinder will now be made, in which both inner and outer expansions will be treated as expansions in $h$, and a matching will be effected as far as $O(h^2)$ corresponding to Riegels's result. Because of the matching, the intuitive inner boundary condition (1.5) will be dispensed with, and it will be found that it only holds good in the limit $h \to 0$, and then only because it accidentally agrees with the condition $w = 0$ for $z = \pm h^*$, which requires by (1.6) that $\int_0^1 H(t)\,dt \equiv 0$, and thus $\int_{-1}^1 \bar{u}_r\,dr = 0$ also (see 1.6, i).

## 2. Formulation of the problem: the solution of $O(1)$

We now suppose that a circular cylinder of radius $a$ is confined between the walls $z' = \pm h^*$ of a Hele-Shaw cell, where the axis of $z'$ is also the axis of the cylinder. We denote the position and velocity vectors of §1 by $\mathbf{r}' = (x', y', z')$, $\mathbf{q}' = (u', v', w')$ respectively, and the pressure by $p'$, and define new outer variables scaled to the dimensions of the problem as follows:

$$\left.\begin{array}{lll}\text{(i)} & \mathbf{r} = (x,y,z) \quad \text{where} & x = x'/a,\ y = y'/a,\ z = z'/h^* = z'/ha,\\ \text{(ii)} & \mathbf{q} = (u,v,w) \quad \text{where} & u = u'/Q,\ v = v'/Q,\ w = w'/Q,\\ \text{(iii)} & p = \mathscr{R}h^2p'/\rho Q^2 \quad \text{where} & \mathscr{R} = \rho Q a/\mu,\ h = h^*/a,\end{array}\right\} \tag{2.1}$$

and, following Riegels, we shall put $\mathscr{R}h^2 = \Lambda$.

As before, $Q$ is the speed of liquid entering the cell in the central plane $z = 0$, and $\rho$ and $\mu$ are the density and viscosity respectively.

At the same time we introduce polar variables $\mathbf{r} = (r, \theta, z)$ where $x = r\cos\theta$, $y = r\sin\theta$, and denote the velocity components corresponding to variations in $r$ and $\theta$ by $\bar{u}$ and $\bar{v}$.

In terms of the Cartesian variables (2.1) the incompressible Navier–Stokes equations can be rearranged (Thompson 1964) to take the following forms (where subscripts denote derivatives):

$$\left.\begin{array}{ll}\text{(i)} & u_{zz} = p_x - h^2\Delta u + \mathscr{R}h^2(uu_x + vu_y + wu_z/h),\\ \text{(ii)} & v_{zz} = p_y - h^2\Delta v + \mathscr{R}h^2(uv_x + vv_y + wv_z/h),\\ \text{(iii)} & w_z = -h(u_x + v_y),\\ \text{(iv)} & p_z = hw_{zz} + h^2\Delta w - \mathscr{R}h^2(huw_x + hvw_y + ww_z),\end{array}\right\} \tag{2.2}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and in the outer region it can be assumed that all variable quantities in (2.2) are $O(1)$. Stokes's equations (1.1) are obtained by taking the limit $h \to 0$ in (2.2) except for (1.1, iv) which requires the additional assumption, not needed here, that $|w| \ll \max(|u|, |v|)$. We also make no assumption at this stage about the size of $\mathscr{R}$.

The boundary conditions are, first, that all the velocity components should vanish at the confining walls and on the obstacle boundary, so that

$$\left.\begin{array}{lll} \text{(i)} & u = v = w = 0 & \text{for} \quad z = \pm 1, \\ \text{(ii)} & \bar{u} = \bar{v} = w = 0 & \text{for} \quad r = 1. \end{array}\right\} \tag{2.3}$$

Also, since in practice the sides of a Hele-Shaw cell must be sealed, say at $y' = \pm L$, we should have $u = v = w = 0$ for $y = \pm L/a$. However, it was shown by Hele-Shaw and Lamb (see Hele-Shaw 1898*a*) that effects due to the seals may be neglected if $L > 15a$ (approximately) and the physical boundary replaced by a condition of uniform Poiseuille flow at infinity, i.e.

$$\left.\begin{array}{l} u = 1-z^2, \quad v = w = 0, \\ p = -2x = -2r\cos\theta \quad \text{at} \quad r \sim \infty. \end{array}\right\} \tag{2.4}$$

In terms of the outer variables, (2.1), Stokes's solution for the leading order in the outer solution may be read off Riegels's solution (1.6) by putting $\Lambda = 0$, and is thus

$$\left.\begin{array}{ll} \text{(i)} & \bar{u} = \bar{u}_0 = (1-z^2)\left(1-\dfrac{1}{r^2}\right)\cos\theta, \\ \text{(ii)} & \bar{v} = \bar{v}_0 = -(1-z^2)\left(1+\dfrac{1}{r^2}\right)\sin\theta, \\ \text{(iii)} & w = w_0 = 0, \quad w/h = w_1 = 0, \\ \text{(iv)} & p = p_0 = -2\left(r+\dfrac{1}{r}\right)\cos\theta, \end{array}\right\} \tag{2.5}$$

which meets all the conditions (2.3, 4) except $\bar{v} = 0$ for $r = 1$.

For the inner region we define certain extra inner variables $X$, $P$, $U$, $V$, $W$ such that

$$\left.\begin{array}{ll} \text{(i)} & r = 1+hX, \quad \text{(ii)} \quad p = P, \\ \text{(iii)} & \mathbf{q} = (U, V, W), \quad \text{using polar directions,} \end{array}\right\} \tag{2.6}$$

and, taking the thickness of the inner region to be $O(h)$, treat both $\partial/\partial X$ and $\partial/\partial z$ as $O(1)$ in the resulting equations of motion.†

This gives, to leading order,

$$\left.\begin{array}{ll} \text{(i)} & P_X = P_z = 0, \qquad \text{(ii)} \quad U_X + W_z = 0, \\ \text{(iii)} & \nabla^2 V = P \quad \text{where} \quad \nabla^2 = \dfrac{\partial^2}{\partial X^2}+\dfrac{\partial^2}{\partial z^2}, \end{array}\right\} \tag{2.7}$$

subject to

(i) $U = V = W = 0$ for $X = 0$,

(ii) as $X \to \infty$, $P$, $U$, $V$, $W$ approach the values of $p$, $\bar{u}$, $\bar{v}$, $w$ at $r = 1$, i.e.

$$P \to -4\cos\theta, \quad V \to -2(1-z^2)\sin\theta, \quad U, W \to 0.$$

† The equations of motion in terms of $X$, $\theta$, $z$ as independent variables are listed in the appendix.

These are the inner equations found by Riegels (1938). Since $P$ is to be independent of $X$ it must retain its outer value $-4\cos\theta$ throughout the inner region, and so (2.7, iii) becomes

$$\nabla^2 V = 4\sin\theta. \tag{2.8}$$

The solution of (2.8) which meets the boundary and matching conditions is

$$V = V_0 = -2\left\{1 - z^2 - 4\sum_0^\infty (-)^n k_n^{-3} e^{-k_n X}\cos k_n z\right\}\sin\theta, \tag{2.9}$$

where $k_n = (n+\frac{1}{2})\pi$, and since the outer $\bar{u}_0$, $w_0$ already meet the inner boundary condition we may take the leading order terms $U_0$, $W_0$ in $U$, $W$ to vanish identically.

The largest term in $X$ in the expansion (2.9) involves $(8/\pi^3)e^{-\frac{1}{2}\pi X}$ which is less than 0·05 when $X \geqslant 2$, and becomes negligible while still $X = O(1)$, thus confirming that for this term the inner region has a thickness that is $O(h)$. The particular value $X = 2$ gives a distance from the obstacle equal to the distance between the walls, which was the boundary-layer thickness predicted by Stokes.

## 3. Terms of order $h$

The form of the outer equations and the Stokes solution suggests that the outer velocity and pressure can be expanded as simple power series in $h$ in the forms

$$\left.\begin{aligned} &\text{(i)} & u &= u_0 + hu_1 + h^2u_2 + \dots,\\ &\text{(ii)} & v &= v_0 + hv_1 + h^2v_2 + \dots,\\ &\text{(iii)} & w &= h^2w_2 + \dots,\\ &\text{(iv)} & p &= p_0 + hp_1 + h^2p_2 + \dots,\end{aligned}\right\} \tag{3.1}$$

where, for example, $u_1 = u_1(x, y, z, \mathscr{R})$, with equivalent expressions for the polar variables.

We shall also assume tentatively that the inner solution can be expanded to give

$$\left.\begin{aligned} &\text{(i)} & U &= hU_1 + h^2U_2 + \dots,\\ &\text{(ii)} & V &= V_0 + hV_1 + h^2V_2 + \dots,\\ &\text{(iii)} & W &= hW_1 + h^2W_2 + \dots,\\ &\text{(iv)} & P &= P_0 + hP_1 + h^2P_2 + \dots,\end{aligned}\right\} \tag{3.2}$$

where, for example, $P_2 = P_2(X, \theta, z, \mathscr{R})$, unless the matching should prove to demand other functions of $h$ in either of these expressions.

By the outer equations of motion (2.2) the terms of $O(h)$ in (3.1) satisfy

$$\left.\begin{aligned} &\text{(i)}\quad u_{1zz} = p_{1x}, && \text{(ii)}\quad v_{1zz} = p_{1y},\\ &\text{(iii)}\quad p_{1z} = 0, && \text{(iv)}\quad w_{2z} = -u_{1x} - v_{1y},\end{aligned}\right\} \tag{3.3}$$

with boundary conditions

$$\left.\begin{aligned} &\text{(i)}\quad u_1 = v_1 = w_2 = 0 \quad \text{for} \quad z = \pm 1,\\ &\text{(ii)}\quad p_1, u_1, v_1, w_2 \to 0 \quad \text{as} \quad r \to \infty.\end{aligned}\right\} \tag{3.4}$$

Thus $p_1, u_1, v_1, w_2$ satisfy the same equations as $p_0, u_0, v_0, w_1$ except that $p_1, u_1 \to 0$ as $r \to \infty$. The solution is therefore

$$\left.\begin{aligned}
&\text{(i)} \quad u_1 = -\tfrac{1}{2}(1-z^2)\,p_{1x}(x,y),\\
&\text{(ii)} \quad v_1 = -\tfrac{1}{2}(1-z^2)\,p_{1y}(x,y),\\
&\text{(iii)} \quad w_2 = \tfrac{1}{2}(z-\tfrac{1}{3}z^3)\,\Delta p_1,
\end{aligned}\right\} \tag{3.5}$$

and by the boundary condition $w_2 = 0$ for $z = \pm 1$, we have

$$\Delta p_1 = \frac{\partial^2 p_1}{\partial x^2} + \frac{\partial^2 p_1}{\partial y^2} \equiv 0, \tag{3.6}$$

giving $w_2 \equiv 0$.

The solution of (3.6) is most conveniently expressed in the polar variables $(r, \theta)$. Since $p_1$ should be an even function of $\theta$, the general form of $p_1(r, \theta)$ must be

$$p_1 = \varphi_0 \log r - \sum_0^\infty \frac{\varphi_n \cos n\theta}{r^n}, \tag{3.7}$$

where the $\varphi_n$ are all constants to be determined later.

In polar variables the solution (3.5) is equivalent to

$$\left.\begin{aligned}
&\text{(i)} \quad \bar{u}_1 = -\tfrac{1}{2}(1-z^2)\,p_{1r}, \quad \text{(ii)} \quad \bar{v}_1 = -\frac{1}{2r}(1-z^2)\,p_{1\theta},\\
&\text{(iii)} \quad w_2 = 0,
\end{aligned}\right\} \tag{3.8}$$

and putting $r = 1 + hX$ in (3.1) we see that for $X \to \infty$ the inner variables $P_1$, $U_1$, $V_1$, $W_1$ must match values

$$\left.\begin{aligned}
&\text{(i)} \quad P_1 = p_1(1,\theta),\\
&\text{(ii)} \quad U_1 = (1-z^2)\,[2X\cos\theta - \tfrac{1}{2}p_{1r}(1,\theta)],\\
&\text{(iii)} \quad V_1 = (1-z^2)\left[2X\sin\theta - \frac{1}{2r}p_{1\theta}(1,\theta)\right],\\
&\text{(iv)} \quad W_1 = 0.
\end{aligned}\right\} \tag{3.9}$$

Substituting (3.2) in the inner equations of motion we obtain terms both of $O(1)$ and $O(h)$ which must satisfy

$$\left.\begin{aligned}
&\text{(i)} \quad P_{1X} = P_{1z} = 0, && \text{(ii)} \quad \nabla^2 V_1 = -XP_{0\theta} + P_{1\theta},\\
&\text{(iii)} \quad \nabla^2 U_1 = P_{2X}, && \text{(iv)} \quad \nabla^2 W_1 = P_{2z},\\
&\text{(v)} \quad U_{1X} + W_{1z} = -V_{0\theta}, && \left(\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial z^2}\right).
\end{aligned}\right\} \tag{3.10}$$

$P_1$ is therefore to be a function of $\theta$ only, and accordingly it must be equal to the quantity $p_1(1, \theta)$ which, of course, has yet to be determined.

By (2.9) we can write $V_{0\theta}$ in explicit form in (3.10, v), giving

$$U_{1X} + W_{1z} = -V_{0\theta} = 2\cos\theta\left[1 - z^2 - 4\sum_0^\infty \frac{(-)^n}{k_n^3}\, e^{-k_n X}\cos k_n z\right],$$

and if we write $p_{1r}(1,\theta) = B(\theta)$ in condition (3.9, ii), so that

$$U_1 \to (1-z^2)\,[2X\cos\theta - \tfrac{1}{2}B(\theta)] \quad \text{as} \quad X \to \infty,$$

then the following transformation is suggested

$$\left.\begin{aligned} &\text{(i)}\ \ U_1 = \psi_z(X,\theta,z) + (1-z^2)\,[2X\cos\theta - \tfrac{1}{2}B(\theta)] \\ &\qquad\qquad + 8\cos\theta \sum_0^\infty \frac{(-)^n}{k_n^3} e^{-k_n X}\cos k_n z, \\ &\text{(ii)}\ \ W_1 = -\psi_X(X,\theta,z), \\ &\text{(iii)}\ \ P_2 = P_2^* - 2X^2\cos\theta + XB(\theta), \end{aligned}\right\} \tag{3.11}$$

by means of which the continuity condition (3.10, v) is identically satisfied.

Substituting in the remaining equations of (3.10) we find

$$\text{(i)}\quad \nabla^2\psi_z = P^*_{2X}, \quad \text{(ii)}\quad \nabla^2\psi_X = -P^*_{2z}. \tag{3.12}$$

Thus after differentiating (3.12, i) with respect to $X$ and (ii) with respect to $z$ there follows

$$\nabla^2 P_2^* = 0,$$

whence, taking the Laplacians of both equations (3.12, i, ii) we have

$$\text{(iii)}\quad \frac{\partial}{\partial X}\nabla^4\psi = \frac{\partial}{\partial z}\nabla^4\psi = 0. \tag{3.12}$$

The matching conditions (3.9) require $\psi_X, \psi_z \to 0$ as $X \to \infty$, and so we can define $\psi \to 0$ as $X \to \infty$ as well, and then (3.12, iii) reduces to the single condition

$$\nabla^4\psi = 0. \tag{3.13}$$

The boundary conditions on $\psi$ are

$$\left.\begin{aligned} &\text{(i)}\quad \psi_X = \psi_z = 0 \quad \text{for} \quad z = \pm 1, \\ &\text{(ii)}\quad \psi_X = 0 \quad \text{for} \quad X = 0, \\ &\text{(iii)}\quad \psi_z = \tfrac{1}{2}(1-z^2)B(\theta) - 8\cos\theta\sum_0^\infty \frac{(-)^n}{k_n{}^4}\cos k_n z \quad \text{for} \quad X = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(cf. (3.11, i))} \\ &\text{(iv)}\quad \psi \to 0 \text{ as } X \to \infty, \quad \text{so that} \\ &\text{(v)}\quad \psi \equiv 0 \quad \text{for} \quad z = \pm 1 \quad \text{by (i).} \end{aligned}\right\} \tag{3.14}$$

These boundary conditions are in fact sufficient in themselves to determine $B(\theta) = p_{1r}(1,\theta)$, for items (i) and (v) require that for all $X$:

$$\begin{aligned} \int_{-1}^{1} z\psi_{zz}\,dz &= [z\psi_z]_{-1}^{1} - \int_{-1}^{1}\psi_z\,dz = -[\psi]_{-1}^{1}, \quad \text{using (i)}, \\ &= 0, \quad \text{using (v)}. \end{aligned} \tag{3.15}$$

This condition must hold in particular when $X = 0$, and applying it to (3.14, iii) we find

$$\int_{-1}^{1}\left\{8\cos\theta\sum_0^\infty \frac{(-)^n}{k_n^3} z\sin k_n z - z^2 B(\theta)\right\}dz = 0,$$

whence
$$B(\theta) = 24\cos\theta \sum_{0}^{\infty} k_n^{-5} = 744\zeta(5)\pi^{-5}\cos\theta, \tag{3.16}$$

where $\zeta(5)$ is a Riemann zeta-function. Thus in (3.7) it follows:

$$\varphi_0 = \varphi_2 = \varphi_3 = \ldots = 0, \quad \text{but} \quad \varphi_1 = 744\pi^{-5}\zeta(5) \simeq 2{\cdot}521,$$

and therefore for $h \to 0$ a term of order $h$ is missing from Riegels's expansion (1.4) which is of lower order in $h$ than the term in $\Lambda = \mathscr{R}h^2$ which he found. The missing terms are given by

$$\text{(i)}\quad p_1 = -\frac{\varphi_1}{r}\cos\theta, \qquad \text{(ii)}\quad \bar{u}_1 = -\tfrac{1}{2}(1-z^2)\frac{\varphi_1\cos\theta}{r^2},$$

$$\text{(iii)}\quad \bar{v}_1 = -\tfrac{1}{2}(1-z^2)\frac{\varphi_1\sin\theta}{r^2}.$$

Riegels' inner condition (1.5) would require of these that

$$\int_{-1}^{1} \bar{u}_1\,dz = 0 \quad \text{for} \quad r = 1,$$

which is not true.

The solutions obtained by Riegels apply, as noted above, in the limit $h \to 0$ with $\mathscr{R}h^2$ finite. However, in order to obtain a visible effect due to secondary flow, Riegels's experiments were in fact carried out with unusually *large* values of $h$, and this suggests that the effect of the terms for $h \neq 0$ ought to have been visible in the photographed streamlines as compared with the plotted streamlines at $\Lambda = 4{\cdot}0$.

Unfortunately Riegels does not record the radius $a$ of the cylinder which was used in his apparatus, but it looks from the photographs and the other published dimensions as if it may have been about 1 cm or less, in which case $h$ would have been about 0·1.

The effect of a non-zero value of $h$ of roughly this size can be quite easily assessed, for upon including $p_1$ in the value of the outer pressure we find that

$$\begin{aligned} p &= p_0 + hp_1 + O(h^2) \\ &= -2\cos\theta\left[r + \frac{1+\frac{1}{2}\varphi_1 h}{r}\right] + O(h^2), \end{aligned}$$

which is the potential for flow past a circular cylinder of radius

$$\{1+\tfrac{1}{2}\varphi_1 h\}^{\frac{1}{2}} \simeq 1 + \tfrac{1}{4}\varphi_1 h$$

or about $1 + 0{\cdot}62h$. Riegels's condition is met at this value of $r$ by the compounded solution
$$\bar{u}^{(1)} = \bar{u}_0 + h\bar{u}_1 \quad \text{since} \quad \bar{u}^{(1)} = 0 \quad \text{for} \quad r = \{1+\tfrac{1}{2}\varphi_1 h\}^{\frac{1}{2}},$$

which shows that there is an interaction between the inner and outer regions which has the effect of defining a *displaced* circular boundary actually in the flow régime. With respect to this boundary the outer flow can still be regarded as a flow past a circular cylinder, but whether the actual streamlines have this appearance or not depends on the behaviour of $U_1$, $V_1$ and $W_1$ at $X \simeq \frac{1}{4}\varphi_1$ since this is a station inside the inner region.

It follows that, to this order, the expansions computed by Riegels can be made to agree with the experimental field simply by increasing the scale of $r$ in the ratio 1·06 : 1 for $h = 0{\cdot}1$. The good agreement observed by Riegels is thus explained since such a small magnification of $r$ would hardly be detectable on a photograph.

Now that $B(\theta)$ is known we can go on to solve for $U_1$, $V_1$ and $W_1$. Equation (3.10, ii) for $V_1$ is not coupled to the others and gives

$$\nabla^2 V_1 = XP_{0\theta} + P_{1\theta} = (\varphi_1 - 4X)\sin\theta,$$

with

$$\text{(i)}\quad V_1 = 0 \quad \text{for} \quad X = 0 \text{ or for } z = \pm 1,$$

$$\text{(ii)}\quad V_1 \to (1-z^2)(2X - \tfrac{1}{2}\varphi_1)\sin\theta \quad \text{as} \quad X \to \infty.$$

Comparison with (2.9) for $V_0$ gives at once

$$\begin{aligned} V_1 &= 2X\sin\theta(1-z^2) + \tfrac{1}{4}\varphi_1 V_0 \\ &= -\tfrac{1}{2}\sin\theta\{(1-z^2)(\varphi_1 - 4X) - \varphi_1 \sum_0^\infty \frac{(-)^n}{k_n^3} e^{-k_n X}\cos k_n z\}, \end{aligned}$$

where, as before, $k_n = (n+\tfrac{1}{2})\pi$.

The values of $U_1$ and $W_1$ depend on the solution of (3.13), i.e. of

$$\nabla^4 \psi = 0$$

subject to the conditions (3.14). For this we shall seek an eigenfunction expansion in the form

$$\psi = \cos\theta \sum_0^\infty \frac{C_r}{K_r^2} e^{K_r X}\phi_r(z), \tag{3.17}$$

where $C_r = A_r + iB_r$, say, are complex constants. Substituting in (3.13) we obtain for each $r$

$$\phi_r^{(\mathrm{iv})} - 2K_r^2\phi_r'' + K_r^4\phi_r = 0,$$

and in order that $\psi_X = \psi_z = 0$ for all $X$ at $z = \pm 1$ we need

$$\phi_r(\pm 1) = \phi_r'(\pm 1) = 0.$$

By the boundary conditions (3.14) $\psi$ should be an odd function of $z$, and under all the conditions we find that

$$\phi_r(z) = (1 + \cos 2K_r)\sin K_r z - 2K_r z\cos K_r z, \tag{3.18}$$

where

$$\sin 2K_r = 2K_r \quad (r = 1, 2, \ldots). \tag{3.19}$$

In order that $\psi \to 0$ as $X \to \infty$, the $K_r$ must all lie in the second quadrant of the complex plane. The approximate distribution of the $K_r$ has been given by Hardy (1902) as

$$2K_r = -(2r + \tfrac{1}{2})\pi + i\log(4r+1)\pi,$$

and precise positions for $1 \leqslant r \leqslant 10$ have been tabulated by Hillman & Salzer (1943) to 6 places of decimals, and a new larger table correct to 9 decimal places for $1 \leqslant r \leqslant 20$ has now been calculated in the course of numerical work associated with the present problem (Thompson 1967).

Although the eigenfunctions $\phi_r(z)$ must evidently satisfy the condition (3.15), i.e.

$$\int_{-1}^{1} z\phi_r''\,dz = 0,$$

they are in no sense mutually orthogonal.

The numerical determination of coefficients $C_r$ to insert in (3.17) therefore constitutes a problem of considerable difficulty, and in particular the most obvious scheme, in which the series (3.17) might be truncated down to, say, the first ten terms and the boundary values fitted by least squares is very highly ill-conditioned.

A workable method has, however, been devised by Gaydon & Shepherd (1964). Here both the individual $\phi_r(z)$ and the given boundary values at $X = 0$ are expanded in terms of orthogonal odd functions $Y_i(z)$ which satisfy the same boundary conditions $Y_i(\pm 1) = Y_i'(\pm 1) = 0$ as the $\phi_r(z)$, and further satisfy a differential equation of the same order, namely

$$Y_i^{(\mathrm{iv})} - \mu_i^4 Y_i = 0. \tag{3.20}$$

If we also impose the normalizing condition

$$\int_{-1}^{1} Y_i^2(z)\,dz = 1,$$

then we find that

$$Y_i = \frac{1}{\surd 2}\left\{\frac{\sin\mu_i z}{\sin\mu_i} - \frac{\sinh\mu_i z}{\sinh\mu_i}\right\},$$

where the $\mu_i$ should be the set of positive solutions of the equation

$$\tan\mu_i = \tanh\mu_i,$$

whose approximate values are $\mu_i = (i+\frac{1}{4})\pi$ $(i = 1, 2, 3, \ldots)$.

In this way an infinite set of linear algebraic equations is generated for the $C_r$, and after inversion these yield the following values of the first 10 coefficients $C_1, C_2, \ldots, C_{10}$ $(C_r = A_r + iB_r)$.

| $r$ | $A_r$ | $B_r$ |
|---|---|---|
| 1 | −0·06942 | −0·00926 |
| 2 | +0·00717 | −0·00995 |
| 3 | −0·00082 | +0·00217 |
| 4 | −0·00012 | −0·00096 |
| 5 | +0·00024 | +0·00048 |
| 6 | −0·00027 | −0·00031 |
| 7 | +0·00041 | +0·00021 |
| 8 | −0·00061 | +0·00013 |
| 9 | −0·00021 | −0·00019 |
| 10 | +0·00001 | −0·00002 |

TABLE 1. Values of $A_r$, $B_r$ found by the method of Gaydon & Shepherd (1964)

By (3.16) the rate of convergence of $V_1$ to the outer value will be the same as that of $V_0$. Values of $U_1$ and $W_1$ are given in tables 2 and 3. Noting that $\mathrm{Re}\,K_1 \simeq -3{\cdot}7$, the largest term in $X$ in $U_1$ will be $e^{-\frac{1}{2}\pi X}/(\frac{1}{2}\pi)^4$ which arises out of the leading term in $X$ in (2.9), so there is no reason to change the original estimate of the thickness of the inner region, i.e. $2h^*$, which was based on the form of (2.9).

| | | $U_1/\cos\theta$ | | | | | Outer value | |
|---|---|---|---|---|---|---|---|---|
| $z\backslash X = 0$ | | 0·1 | 0·4 | 0·6 | 1·0 | 2·0 | 1·0 | 2·0 |
| 1·0 | $< 10^{-4}$ | 0·000 | 0·000 | 0·000 | 0·000 | 0·000 | 0·000 | 0·000 |
| 0·9 | | 0·002 | 0·034 | 0·074 | 0·184 | 0·529 | 0·14 | 0·52 |
| 0·8 | | 0·004 | 0·068 | 0·145 | 0·353 | 1·00 | 0·27 | 0·94 |
| 0·7 | | 0·007 | 0·101 | 0·211 | 0·503 | 1·42 | 0·38 | 1·40 |
| 0·6 | | 0·010 | 0·131 | 0·270 | 0·636 | 1·79 | 0·48 | 1·76 |
| 0·5 | | 0·012 | 0·157 | 0·321 | 0·749 | 2·09 | 0·56 | 2·06 |
| 0·4 | | 0·013 | 0·179 | 0·363 | 0·842 | 2·35 | 0·63 | 2·31 |
| 0·3 | | 0·015 | 0·195 | 0·396 | 0·915 | 2·54 | 0·68 | 2·50 |
| 0·2 | | 0·016 | 0·207 | 0·419 | 0·967 | 2·68 | 0·72 | 2·64 |
| 0·1 | | 0·016 | 0·214 | 0·434 | 0·999 | 2·77 | 0·74 | 2·72 |
| 0·0 | | 0·016 | 0·217 | 0·438 | 1·009 | 2·80 | 0·75 | 2·75 |

TABLE 2. Values of $U_1/\cos\theta$ found by summing the values of the first ten terms of (3.17), compared with the outer terms of $O(h)$ at $X = 1{\cdot}0$ and $X = 2{\cdot}0$

| $z\backslash X = 0$ | | 0·1 | 0·4 | 0·6 | 1·0 | 2·0 |
|---|---|---|---|---|---|---|
| 1·0 | $< 10^{-5}$ | 0·000 | 0·000 | 0·000 | 0·000 | 0·000 |
| 0·9 | | 0·002 | 0·002 | 0·001 | 0·000 | $< 10^{-4}$ |
| 0·8 | | 0·006 | 0·006 | 0·003 | 0·001 | |
| 0·7 | | 0·009 | 0·010 | 0·006 | 0·002 | |
| 0·6 | | 0·012 | 0·014 | 0·009 | 0·003 | |
| 0·5 | | 0·012 | 0·015 | 0·010 | 0·003 | |
| 0·4 | | 0·011 | 0·015 | 0·010 | 0·003 | |
| 0·3 | | 0·010 | 0·013 | 0·009 | 0·003 | |
| 0·2 | | 0·007 | 0·010 | 0·007 | 0·002 | |
| 0·1 | | 0·004 | 0·005 | 0·004 | 0·001 | |
| 0·0 | | 0·000 | 0·000 | 0·000 | 0·000 | |

TABLE 3. Values of $W/\cos\theta$ found by summing the values of the first ten terms of (3.17)

## 4. Terms of order $h^2$

The terms of $O(h^2)$ in the outer expansion (3.1) satisfy equations

$$\left.\begin{aligned} &\text{(i)} \quad u_{2zz} = p_{2x} + \mathscr{R}(u_0 u_{0x} + v_0 u_{0y}),\\ &\text{(ii)} \quad v_{2zz} = p_{2y} + \mathscr{R}(u_0 v_{0x} + v_0 v_{0y}),\\ &\text{(iii)} \quad p_{2z} = 0,\\ &\text{(iv)} \quad w_{3z} = -u_{2x} - v_{2y}. \end{aligned}\right\} \tag{4.1}$$

Since we have

$$u_0 = -\tfrac{1}{2}(1-z^2)p_{0x}, \quad v_0 = -\tfrac{1}{2}(1-z^2)p_{0y},$$

it follows that

$$u_{2zz} = \frac{\partial}{\partial x}\{p_2 + \tfrac{1}{8}\mathscr{R}(1-2z^2+z^4)(\nabla p_0)^2\},$$

where $(\nabla p_0)^2 = p_{0x}^2 + p_{0y}^2$, or, by the condition $u_2 = 0$ for $z = \pm 1$,

$$\left.\begin{aligned} &\text{(i)} \quad u_2 = -\tfrac{1}{2}(1-z^2)p_{2x} + \mathscr{R}\left(\frac{z^6}{120} - \frac{z^4}{24} + \frac{z^2}{8} - \frac{11}{120}\right)\frac{\partial}{\partial x}\tfrac{1}{2}(\nabla p_0)^2,\\ &\text{and similarly}\\ &\text{(ii)} \quad v_2 = -\tfrac{1}{2}(1-z^2)p_{2y} + \mathscr{R}\left(\frac{z^6}{120} - \frac{z^4}{24} + \frac{z^2}{8} - \frac{11}{120}\right)\frac{\partial}{\partial y}\tfrac{1}{2}(\nabla p_0)^2,\\ &\text{whence by (4.1, iv)}\\ &\text{(iii)} \quad w_3 = \tfrac{1}{2}(z - \tfrac{1}{3}z^3)\,\Delta p_2 + \mathscr{R}\left(\frac{z^7}{840} - \frac{z^5}{120} + \frac{z^3}{24} - \frac{11z}{120}\right)\Delta\tfrac{1}{2}(\nabla p_0)^2. \end{aligned}\right\} \tag{4.2}$$

Since $w_3 = 0$ for $z = \pm 1$, we now have by (4.2, iii)

$$\Delta p_2 = -\tfrac{6}{35}\mathscr{R}\,\Delta\tfrac{1}{2}(\nabla p_0)^2,$$

and solving this in polar variables with $p_0$ given as

$$p_0 = -2\left(r + \frac{1}{r}\right)\cos\theta$$

by (2.5, iv), we arrive at the following form for $p_2$:

$$p_2 = \mathscr{R}\left\{\varpi(r,\theta) - \frac{12}{35}\left(1 - \frac{2}{r^2}\cos 2\theta + \frac{1}{r^4}\right)\right\}, \tag{4.3}$$

where $\varpi$ is an undetermined even function of $\theta$ satisfying

$$\Delta\varpi = 0,$$

and since $u_2 \to 0$ as $r \to \infty$, we need $\partial w/\partial r \to 0$ as $r \to \infty$. Hence evidently the general solution for $\varpi(r,\theta)$ is

$$\varpi = \varphi_0' \log r + \sum_1^\infty \varphi_n' \frac{\cos n\theta}{r^n}, \tag{4.4}$$

the $\varphi_n'$ being constants.

Expanding (4.2, i, ii) in powers of $h$ about $r = 1$, the terms of order $h^2$ show that the matching values of $U_2$, $V_2$, $W_2$ for $X \to \infty$ are to be

$$\left.\begin{aligned} &\text{(i)} \ U_2 = (1-z^2)(\varphi_1 X - 3X^2)\cos\theta + \mathscr{R}\{H(z)(\cos 2\theta - 1) - \tfrac{1}{2}(1-z^2)\,\varpi_r(1,\theta)\},\\ &\text{(ii)} \ V_2 = (1-z^2)(\varphi_1 X - 3X^2)\sin\theta + \mathscr{R}\{H(z)\sin 2\theta - \tfrac{1}{2}(1-z^2)\,\varpi_r(1,\theta)\},\\ &\text{(iii)} \ W_2 = 0, \end{aligned}\right\} \tag{4.5}$$

where, as in (1.6),

$$H(z) = \frac{z^6}{15} - \frac{z^4}{3} + \frac{11}{35}z^2 - \frac{1}{21}.$$

The inner equations of motion show that

$$\left.\begin{aligned} &\text{(i)} \quad \nabla^2 U_2 = P_{3X} - \mathscr{R}V_0^2 + 2V_{0\theta} - U_{1X},\\ &\text{(ii)} \quad \nabla^2 W_2 = P_{3z} - W_{1z},\\ &\text{(iii)} \quad U_{2X} + W_{2z} = -U_1 + XV_{0\theta} - V_{1\theta}, \end{aligned}\right\} \tag{4.6}$$

and there is an uncoupled equation for $V_2$ which is not quoted.

The set of equations (4.6) will obviously be very difficult indeed to solve, but our purpose will be sufficiently served if we can find another stream function as in §3 whose boundary values determine $\varpi(r,\theta)$, since this will determine the outer solution to the order found by Riegels without recourse to the intuitive inner condition (1.5). The appropriate transformation this time is

$$\left.\begin{aligned}
&\text{(i)}\quad U_2 = \psi_z + (1-z^2)(\varphi_1 X - 3X^2)\cos\theta \\
&\qquad\quad + \mathscr{R}\{H(z)(\cos 2\theta - 1) - \tfrac{1}{2}(1-z^2)\varpi_r(1,\theta)\} \\
&\qquad\quad + \cos\theta\left\{4(\varphi_1 - 4X)\sum_0^\infty [(-)^n e^{-k_n X}\cos k_n z]/k_n^4\right. \\
&\qquad\quad \left. + \operatorname{Re}\sum_1^\infty \frac{C_r}{K_r^3} e^{K_r X}\phi_r'(z)\right\}, \\
&\text{(ii)}\quad W_2 = -\psi_X.
\end{aligned}\right\} \tag{4.7}$$

Combining (4.6, i, ii, iii) we find that

$$\nabla^2 P_3 = 2\mathscr{R} V_0 V_{0X} - V_{0\theta X} + \nabla^2 U_1 + X\nabla^2 V_{0\theta} - \nabla^2 V_{1\theta},$$

and so taking Laplacians of (4.6, i, ii) it follows that

$$\left.\begin{aligned}
&\text{(i)}\quad \nabla^4\psi_z = -2\mathscr{R}(V_{0z}^2 + V_0 V_{0zz}) - V_{0\theta XX} - \nabla^2(2U_{1X} - 2V_{0\theta} + V_{1\theta X}) \\
&\qquad\qquad + X\nabla^2 V_{0\theta X} - 12\cos\theta + 16\mathscr{R}(1-3z^2)\sin^2\theta \\
&\qquad\quad = F_1, \quad \text{say}, \\
&\text{(ii)}\quad -\nabla^4\psi_X = 2\mathscr{R}(V_{0z}V_{0x} + V_0 V_{0Xz}) - V_{0\theta Xz} - \nabla^2(U_{1z} + V_{1\theta z} + W_{1X}) \\
&\qquad\qquad + X\nabla^2 V_{0\theta z} \\
&\qquad\quad = F_2, \quad \text{say}.
\end{aligned}\right\} \tag{4.8}$$

It is a simple exercise in algebra to show that $F_{1X} + F_{2z} = 0$ as well as $F_1, F_2 \to 0$ as $X \to \infty$. Hence $F_1 dz - F_2 dX$ is a perfect differential and so there exists a single equation in the form

$$\nabla^4\psi = F(X,\theta,z,\mathscr{R}), \tag{4.9}$$

which is equivalent to (4.8, i, ii).

The function $\psi(X,\theta,z,\mathscr{R})$ must by (4.5) and the condition $U_2 = W_2 = 0$ for $X = 0$ satisfy the following boundary conditions

$$\left.\begin{aligned}
&\text{(i)}\quad \psi_X = \psi_z = 0 \quad \text{for} \quad z = \pm 1, \\
&\text{(ii)}\quad \psi_X, \psi_z \to 0 \quad \text{as} \quad X \to \infty, \\
&\text{(iii)}\quad \psi_X = 0 \quad \text{at} \quad X = 0, \quad \text{and also} \\
&\text{(iv)}\quad \psi_z = \mathscr{R}\{\tfrac{1}{2}(1-z^2)\varpi_r(1,\theta) - H(z)(\cos 2\theta - 1)\} \\
&\qquad - \cos\theta\left\{4\varphi_1\sum_0^\infty \frac{(-)^n}{k_n^4}\cos k_n z + \operatorname{Re}\sum_0^\infty \frac{C_r}{K_r^3}\phi_r'(z)\right\},
\end{aligned}\right\} \tag{4.10}$$

and once again the condition (3.15) is applicable to give

$$\int_{-1}^{1} z\psi_{zz}\,dz = 0 \quad \text{for all } X,$$

which requires, when $X = 0$, that

$$\varpi_r(1,\theta) = \frac{8}{3\mathscr{R}}\varphi_1\cos\theta\sum_0^\infty\frac{1}{k_n^5} \quad [k_n = (n+\tfrac{1}{2})\pi],$$

or, by (3.16),

$$\varpi(r,\theta) = -\varphi_1^2\cos\theta/9\mathscr{R}r.$$

Thus, substituting for $\varpi(r,\theta)$ in (4.2) and expressing the results in polar variables, we find

$$\left.\begin{aligned}
&\text{(i)} \quad \bar{u}_2 = -\tfrac{1}{18}\varphi_1^2(1-z^2)\frac{\cos\theta}{r^2} + \mathscr{R}H(z)\left(\frac{\cos 2\theta}{r^3} - \frac{1}{r^5}\right),\\
&\text{(ii)} \quad \bar{v}_2 = -\tfrac{1}{18}\varphi_1^2(1-z^2)\frac{\sin\theta}{r^2} + \mathscr{R}H(z)\frac{\sin 2\theta}{r^3},\\
&\text{(iii)} \quad w_2 = 0, \quad w_3 = -\frac{4}{r^6}\int_0^z H(z)\,dz,\\
&\text{(iv)} \quad p_2 = -\tfrac{1}{9}\varphi_1^2\frac{\cos\theta}{r} - \frac{12}{35\mathscr{R}}\left(1 - \frac{2\cos 2\theta}{r^2} + \frac{1}{r^4}\right).
\end{aligned}\right\} \quad (4.11)$$

From (4.11) it is clear that once again Riegels's inner condition is not met since

$$\int_{-1}^1 \bar{u}_2\,dz \neq 0 \quad \text{for} \quad r = 1.$$

However, since by (4.3)

$$\int_{-1}^1 H(z)\,dz = 0,$$

it becomes correct in the limit $h \to 0$, so the terms found by Riegels give the correct coefficients of $\mathscr{R}$ in (3.1).

The effect of the term $\varpi(r,\theta)$ in $p_2$ is to give a pressure

$$\begin{aligned}
p &= p_0 + hp_1 + h^2p_2 + O(h^3)\\
&= -2\cos\theta\left\{r + \frac{1+\frac{1}{2}h\varphi_1 + \frac{1}{18}h^2\varphi_1^2}{r}\right\} - \tfrac{12}{35}\Lambda\left\{1 - \frac{\cos 2\theta}{r^2} + \frac{1}{r^4}\right\} + O(h^3), \quad (4.12)
\end{aligned}$$

which displaces the circular boundary upon which the outer flow is apparently based a little further out, but again to this order the field of streamlines plotted by Riegels can be made to give the actual flow field by a small linear increase in the scale of $r$.

Nevertheless some misgivings must still be felt about applying the expansions (3.1) to such a large value of $\Lambda$ as 4·0, and the apparent occurrence of a boundary-layer separation in Riegels's photographed flow would suggest that these misgivings are well founded.

## Appendix. The inner equations of motion

*r* direction

$$\mathscr{R}h^2\left\{\frac{U}{h}U_X + \frac{V}{1+hX}U_\theta + \frac{W}{h}U_z - \frac{V^2}{1+hX}\right\} + \frac{1}{h}P_X$$
$$= \frac{\partial}{\partial X}\left[\frac{1}{1+hX}\frac{\partial}{\partial X}\{U(1+hX)\}\right] + \frac{h^2}{(1+hX)^2}U_{\theta\theta} + U_{zz} - \frac{2h^2}{(1+hX)^2}V_\theta.$$

$\theta$ direction

$$\mathcal{R}h^2\left\{\frac{U}{h}V_X+\frac{V.V_\theta}{1+hX}+\frac{W}{h}V_z+\frac{UV}{1+hX}\right\}+\frac{1}{1+hX}P_\theta$$
$$=\frac{\partial}{\partial X}\left[\frac{1}{1+hX}\frac{\partial}{\partial X}\{V(1+hX)\}\right]+\frac{h^2}{(1+hX)^2}V_{\theta\theta}+V_{zz}+\frac{2h^2}{(1+hX)^2}U_\theta.$$

$z$ direction

$$\mathcal{R}h^2\left\{\frac{U}{h}W_X+\frac{V}{1+hX}W_\theta+\frac{W}{h}W_z\right\}+\frac{1}{h}P_z$$
$$=\frac{1}{1+hX}\frac{\partial}{\partial X}\left[(1+hX)\frac{\partial W}{\partial X}\right]+\frac{h^2}{(1+hX)^2}W_{\theta\theta}+W_{zz}.$$

Continuity

$$\frac{\partial}{\partial X}[(1+hX)U]+hV_\theta+(1+hX)\frac{\partial W}{\partial z}=0.$$

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